【極式的運算性質】

若複數 $Z_1=r_1(\cos\theta_1+i\sin\theta_1)$ 、 $Z_2=r_2(\cos\theta_2+i\sin\theta_2)$ ， r_1 、 $r_2\in\mathbb{R}$ ，則

(1) $Z_1Z_2=r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))$ [絕對值相乘，輻角相加]

(2) $Z_1^{-1}=r_1^{-1}(\cos(-\theta_1)+i\sin(-\theta_1))$

(3) $\dfrac{Z_1}{Z_2}=\dfrac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))$ [絕對值相除，輻角相減]

[證明]

(1)
$$\begin{aligned}Z_1Z_2&=r_1(\cos\theta_1+i\sin\theta_1)\times r_2(\cos\theta_2+i\sin\theta_2)\\&=r_1r_2[(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)]\\&=r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))\end{aligned}$$

(2)
$$\begin{aligned}Z_1^{-1}&=\frac{1}{Z_1}=\frac{1}{r_1(\cos\theta_1+i\sin\theta_1)}\\&=\frac{\cos\theta_1-i\sin\theta_1}{r_1(\cos\theta_1+i\sin\theta_1)(\cos\theta_1-i\sin\theta_1)}=\frac{1}{r_1}\times\frac{\cos\theta_1-i\sin\theta_1}{\cos^2\theta_1+\sin^2\theta_1}\\&=\frac{1}{r_1}(\cos\theta_1-i\sin\theta_1)=r_1^{-1}(\cos(-\theta_1)+i\sin(-\theta_1))\end{aligned}$$

(3)
$$\begin{aligned}\frac{Z_1}{Z_2}&=Z_1Z_2^{-1}=r_1(\cos\theta_1+i\sin\theta_1)\times r_2^{-1}(\cos(-\theta_2)+i\sin(-\theta_2))\\&=r_1r_2^{-1}(\cos(\theta_1+(-\theta_2))+i\sin(\theta_1+(-\theta_2)))=\frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))\end{aligned}$$

【棣美弗定理】

若複數 $Z=r(\cos\theta+i\sin\theta)$，$r\in\mathbb{R}$、$n\in\mathbb{Z}$，則 $Z^n=r^n(\cos n\theta+i\sin n\theta)$ [絕對值 n 方，輻角 n 倍]

[證明]

(1) 當 $n\in\mathbb{N}$ 時，使用數學歸納法

① $Z^1=r^1(\cos\theta+i\sin\theta)$ $\therefore$ $n=1$ 時，命題成立

② 假設 $n=k$ 時，命題成立，亦即 $Z^k=r^k(\cos k\theta+i\sin k\theta)$

$$Z^{k+1}=Z^kZ=r^k(\cos k\theta+i\sin k\theta)\times r(\cos\theta+i\sin\theta)$$

$$=r^{k+1}(\cos(k\theta+\theta)+i\sin(k\theta+\theta))$$

$$=r^{k+1}(\cos(k+1)\theta+i\sin(k+1)\theta)$$

$\therefore$ $n=k+1$ 時，命題也成立

由①、②故得證

(2) 當 $n=0$ 時，$Z^0=r^0(\cos 0+i\sin 0)=1$，命題成立

(3) 當 n 為負整數時，令 $n=-m$，$m\in\mathbb{N}$

$Z^n=Z^{-m}=(Z^m)^{-1}=r^{-m}(\cos(-m\theta)+i\sin(-m\theta))=r^n(\cos n\theta+i\sin n\theta)$，命題成立